Damped bistable system driven by colored noise: A digital simulation study

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We present digital computer simulation results of stochastic differential equations associated with a damped bistable potential. The mean first passage time (MFPT) and the stationary probability density function (SPDF) for the case of a damped bistable potential with white and colored noise are computed. For the white-noise-driven bistable potential, we compute the MFPT and compare it with that of earlier theoretical results. For the colored-noise-driven bistable potential, we compute the SPDF, which compares favorably with already available analog simulation results. We also compute the MFPT for the colored-noise case. Our simulation results are expected to act as a benchmark for comparing future theoretical results of damped stochastic system driven by colored noise. [S1063-651X(98)03209-7]

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I. INTRODUCTION

Kramers's problem is a long-standing problem in the area of stochastic modeling of dynamical systems. Kramers's problem is concerned with the computation of the escape rate of a Brownian particle from a metastable state subject to a noise and with a finite damping. The problem was stated and solved under certain restricted conditions by Kramers [1]. The phenomenon of noise activated escape from a metastable state plays a central role in many areas of physics, chemistry, and biology. In condensed-matter physics, Kramers's problem is encountered in a variety of phenomena ranging from superionic conduction [2], diffusion of atoms at crystal surfaces [3], Josephson junction theory [4], and a phase-locked loop device [4] to a driven Ge photoconductor [5]. The ring-laser gyroscope [6], dye laser [7,8], optical logic, and optical computing devices [9] are examples of Kramers's problem in optical physics. Another interesting variety of Kramers's problem is the transport phenomenon in complex systems as it occurs in glasses [10,11] and proteins [12,13]. For a recent review on Kramers's problem, see Refs. [14,15].

Noise activated escape occurs via two mechanisms, namely, thermal [16] and quantum [17] activation. In both these classical and quantum regimes, Kramers's problem has been a subject of active research.

In the classical limit, the Langevin equation [4,18] describing the Brownian motion of a particle of unit mass in a one-dimensional potential $U(x)$ is

$$\ddot{x} = -\gamma\dot{x} - \frac{dU(x)}{dx} + \xi(t), \tag{1}$$

where x is the position of the Brownian particle, γ is the damping coefficient, and $\xi(t)$ is the noise driving the system. In Eq. (1) overdots represent derivatives with respect to time.

For the widely discussed case of the bistable potential, $U(x)$ takes the form

$$U(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \tag{2}$$

Henceforth we take $U(x)$ to be a bistable potential. The system described by Eq. (1) can be made two dimensional:

$$\dot{x} = v, \tag{3a}$$

$$\dot{v} = x - x^3 - \gamma v + \xi(t), \tag{3b}$$

where v is the velocity of the Brownian particle. Supposing that $\xi(t)$ is Gaussian white noise, we have [19]

$$\langle \xi(t)\xi(s)\rangle = 2D\,\delta(t-s),$$

where D is the noise strength, $\delta(t)$ is the Dirac delta function, and angular brackets represent ensemble averaging. This is the case of a white-noise-driven damped bistable potential. Considerable progress has been made in this area [14,15,20–25].

When the noise $\xi(t)$ is colored, in particular, if $\xi(t)$ is an Ornstein-Uhlenbeck (OU) process (i.e., exponentially correlated) as defined by

$$\langle \xi(t)\xi(s)\rangle = \frac{D}{\tau}\exp(-|t-s|/\tau),$$

Eq. (3) becomes a three-dimensional Markovian process and takes the form

$$\dot{x} = v, \tag{4a}$$

$$\dot{v} = x - x^3 - \gamma v + \xi(t), \tag{4b}$$

$$\dot{\xi} = -(1/\tau)\xi + \frac{1}{\tau}\,\eta(t), \tag{4c}$$

where τ is the noise correlation time and $\eta(t)$ is the Gaussian white noise of strength D. Equation (4) is the general case of a damped bistable potential driven by colored noise. Because of the lack of a detailed balance symmetry, the nonlinear three-dimensional Markovian process given by Eq. (4) cannot be solved in an exact analytical way [4].

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By invoking the limit $\gamma\to\infty$, v can be adiabatically eliminated and one gets the case of an overdamped bistable potential driven by colored noise. The governing equations can then be written as

$$\dot{x}=x-x^3+\xi(t), \tag{5a}$$

$$\dot{\xi}=-(1/\tau)\xi+\frac{1}{\tau}\ \eta(t), \tag{5b}$$

where $\eta(t)$ is the Gaussian white noise.

Recently, there has been considerable progress in solving Eq. (5) [26] (see Refs. [35, 36]). However, the problem still remains to be solved for the general case of an arbitrary damping coefficient along with a noise of finite correlation time [i.e., Eq. (4)]. No satisfactory theory is currently available for the problem of a damped stochastic system driven by colored noise. In such a condition, analog [27] and digital simulations are the only viable ways of getting a first-hand clue to the statistical properties of processes described by Eq. (4). In this paper, we present digital simulation results of a stochastic system with finite damping driven by white and colored noise.

The paper is organized as follows. In Sec. II we discuss the digital simulation procedure. In Sec. III we present the simulation results for the mean first passage time (MFPT) of a Brownian particle in a damped bistable potential driven by white noise. Our MFPT results are compared with analytical results of earlier theories [1,15] and the matrix continued fraction (MCF) results [21]. Section IV is devoted to digital simulation studies [computation of the stationary probability density function (SPDF) and the MFPT] of the damped bistable potential driven by colored noise. The SPDF is computed and compared with the corresponding results of analog simulation [28]. Next we present the results for the MFPT. We then compare our MFPT results with that of an *ad hoc* formula [which is formulated by us and is supposed to be valid for computing the MFPT for the case of Eq. (4)]. Section V contains our conclusions.

II. DIGITAL SIMULATION PROCEDURE

Several algorithms have been proposed recently to integrate stochastic differential equations [29–31]. Our simulation is based on the second-order algorithm proposed by Fox in Ref. [31]. The algorithm presented by Fox [31] is genuinely second order for both the deterministic and stochastic portions. An improved but complex algorithm has been proposed by Milshtein and Tret'yakov [32]. We have adapted Fox's algorithm and applied it to integrate our multidimensional stochastic differential equations (3) and (4). The Box-Muller algorithm [33] has been used to generate Gaussian white noise needed in our simulation.

We want to compare our simulation results with theoretical prediction of Kramers [1] and Mel'nikov and Meshkov [15,20]. These theories demand that $\Delta U \gg k_B T$, where ΔU denotes the height of the potential barrier, which is $\frac{1}{4}$ in our case, k_B is the Boltzmann constant, and T is the absolute temperature of the surrounding heat bath. In our simulation, we have kept $\Delta U/k_B T=2.5$. This relation is equivalent to fixing $D/\gamma=0.1$ (since $D=\gamma k_B T$), where D is the strength

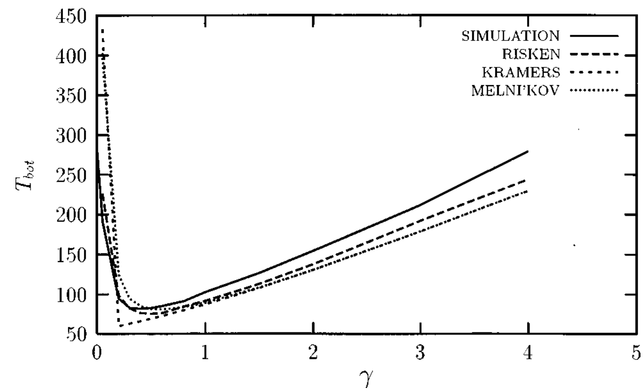

FIG. 1. $T_{\rm bot}$ as a function of γ for the damped bistable potential driven by white noise with $\Delta U/k_B T=2.5$.

of the white noise and γ is the damping coefficient. For $\gamma=0$, the value of D cannot be chosen to satisfy the above relationship. Therefore, in our simulation, to obtain $T_{\rm bot}$ with $\gamma=0$, we have chosen a small value of D (0.001). We have used a small integration step size (the value being less than or equal to 0.01) to achieve better convergence. However, it has been noticed that increasing the step size even to the order of 0.1 does not create any remarkable deviation in the results. This establishes the superiority of the second-order algorithm over first-order ones. In our simulation procedure using colored noise, we have kept $h/\tau \ll 1$ in order to ensure the stability of the algorithm. Here h is the integration step size.

For both white and colored noise, we have computed the MFPT for the Brownian particle in the bistable potential starting from the bottom of one well $(x=-1)$ to reach the bottom of the other well $(x=+1)$. This time is called the $T_{\rm bot}$. $T_{\rm bot}$ has been calculated by averaging the first passage times (FPTs) over 3000 iterations for each choice of parameter value. The averaging keeps the statistical error in our simulation results below $\pm 5\%$.

The SPDF is computed as follows. First, the MFPT is computed using the above-mentioned procedure. The x axis ranging from -4 to $+4$ is divided into small intervals Δx of size 0.025. Starting from an arbitrary value of x, we follow the stochastic trajectory of x by simulating Eq. (3) [or Eq. (4) as the case may be] for a time interval of 5000–15 000 times the MFPT. For $D=0.32$, we have taken 5000 times the MFPT and for $D=1.0$ we have taken 15 000 times the MFPT. A counter is maintained for each interval Δx and is initially set to 0 before the simulation is started. The x trajectory is followed by recomputing x for every integration step size. The respective counter is incremented whenever x falls within the given interval. Finally, we normalize the counts to get the SPDF.

III. DAMPED BISTABLE SYSTEM DRIVEN BY WHITE NOISE

In this section we present the results for $T_{\rm bot}$ obtained by simulating the damped bistable potential driven by white noise [Eq. (3)]. Our results for $T_{\rm bot}$ for various values of γ are shown in Fig. 1. In Fig. 1 we also compare our results for $T_{\rm bot}$ with the inverse of escape rate proposed by Kramers [1], Mel'nikov and Meshkov [15,20], and Risken and Voigtlander [21]. Regarding Kramers's result, we have used both formulas proposed by Kramers, namely, the formula valid for the underdamped case and the formula valid for the moderate to strong friction case.

It is observed from the results plotted in Fig. 1 that with increasing γ, T_{bot} undergoes a turnover from an inverse trend ($T_{\text{bot}} \propto 1/\gamma$) to a linear behavior ($T_{\text{bot}} \propto \gamma$). This was noted by Kramers [1] and subsequently supported by other results [15,20,21]. The MFPT in our simulation has a finite value for $\gamma = 0$ for finite noise strength D and approaches infinity as $\gamma \to \infty$. The MFPT has a minimal value intermediate between these two limits.

Our digital simulation results agree with the MCF results of Risken and Voigtlander better than the theoretical results in the whole damping range. On comparing the two theoretical predictions in the underdamped region ($\gamma/\omega_b < 1$, where ω_b is the frequency of oscillations at the barrier top), Mel'nikov's results are found to be in better agreement with our simulation results than Kramers's results. However, in the overdamped region ($\gamma/\omega_b > 1$), Mel'nikov's results coincide with Kramers's result and both theories underestimate our T_{bot}.

IV. DAMPED BISTABLE SYSTEM DRIVEN BY COLORED NOISE

A. SPDF

We compare the SPDF obtained through our digital simulation with that of the analog simulation carried out by Fronzoni *et al.* [28]. Fronzoni *et al.* have done an analog simulation of the damped bistable oscillator driven by colored noise and have reported the values for R (the ratio of the maximum to the minimum in the SPDF curve) for different values of D, γ, and τ. We have performed digital simulation and have computed the SPDF for the same sets of D, γ, and τ as used by Fronzoni *et al.* In Table I we compare the values of R obtained through our digital simulation with those of Fronzoni *et al.* [28]. We observe a fairly good coincidence of the values of R obtained through digital simulation with those of analog simulation.

TABLE I. Comparison of digital and analog simulation results for R (the ratio of the maximum to the minimum of the SPDF curve) for different values of D (noise strength), γ (damping coefficient), and τ (noise correlation time).

D	γ	τ	R by our digital simulation	R by analog simulation [28]
0.32	1.0	0.63	3.66	3.20
0.32	1.0	1.17	6.72	6.80
0.32	1.0	2.14	16.22	19.00
1.0	1.0	1.00	2.41	2.90
1.0	1.0	0.50	1.70	1.93
1.0	1.0	0.10	1.33	1.44

In Figs. 2 and 3 values of the SPDF $P(x)$ calculated by our digital simulation are plotted against x for various values of D, γ, and τ. The behavior of the SPDF with a change in D, γ, and τ is in qualitative agreement with the conclusions drawn from the analog simulation [28]. It is observed that with increasing τ at fixed D and γ, the spreading of the distribution is reduced and the distribution becomes more peaked.

In Fig. 4 values of the SPDF $P(x)$ are plotted against x for various values of γ with fixed D and τ. It is seen again that the spreading of the distribution is reduced and the distribution becomes more peaked with increasing γ. Thus it is observed that a change of γ affects the SPDF curve in a way qualitatively similar to a change of τ.

B. MFPT

In Fig. 5 we plot T_{bot} vs τ for various values of γ with $D = 0.5$. Similarly to the widely discussed overdamped case, the MFPT varies exponentially with τ for all values of damping coefficient.

In Fig. 6 we plot T_{bot} vs γ for various values of τ in the moderate to strong friction limit. In Fig. 7 we plot T_{bot} vs γ for various values of τ for the weak damping case. We keep D at 0.5 throughout our simulation reported in Figs. 6 and 7. It is observed from these plots that T_{bot} increases exponentially with an increase in γ. Therefore, we conclude that an increase in τ or an increase in γ has qualitatively the same effect on T_{bot}.

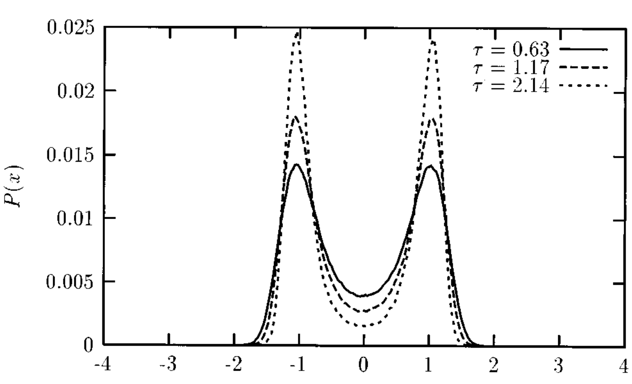

FIG. 2. SPDF for the damped bistable potential driven by OU noise for various τ with $D = 0.32$ and $\gamma = 1.0$.

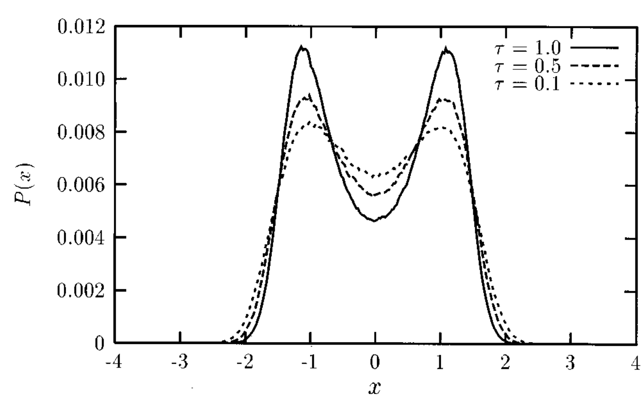

FIG. 3. SPDF for the damped bistable potential driven by OU noise for various τ with $D = 1.0$ and $\gamma = 1.0$.

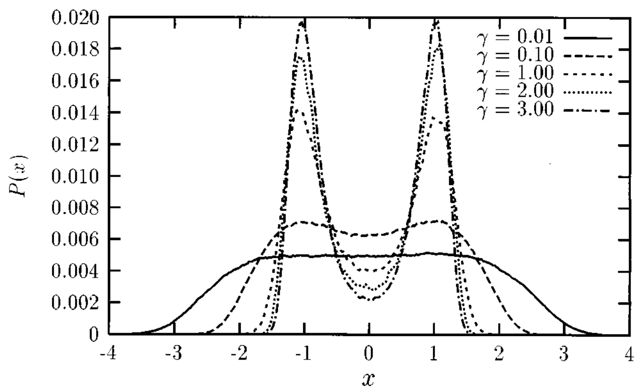

FIG. 4. SPDF for the damped bistable potential driven by OU noise for various γ with $D=0.5$ and $\tau=1.0$.

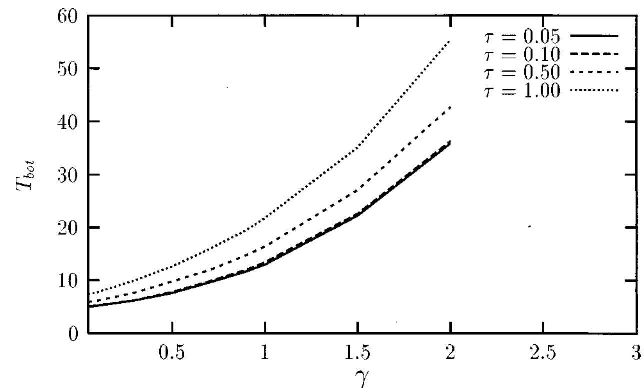

FIG. 6. $T_{\rm bot}$ vs γ for the damped bistable potential driven by OU noise with $D=0.5$ for various τ in the moderate to strong friction limit.

Notice from Figs. 6 and 7 (also see Fig. 5) that we do not observe any turnover in the value of $T_{\rm bot}$ on changing γ from the underdamped to the overdamped limit. This has been observed in the entire range of τ explored by us: $\tau=0.05-5$ (indeed, we have even gone up to $\tau=0.01$). However, the turnover behavior of the MFPT for the white noise case is very well established (see Fig. 1). It is perplexing why the MFPT changes its behavior from the white-noise case to the colored-noise case (even for low values of correlation time).

We are unable to compare our simulation results with those of any theoretical results as no theoretical formula currently exists (to our knowledge) to calculate the MFPT of a damped bistable potential driven by colored noise. On the other hand, much theoretical work has been done for the underdamped white-noise case and the overdamped colored-noise case. In order to evaluate the validity of the theories proposed in the above-mentioned cases for damped stochastic system driven by colored noise, we formulate an *ad hoc* theoretical expression as follows. To arrive at an approximate formula valid for small correlation time and weak damping, we just plug the prefactor term present in the formula for the MFPT proposed by Hanggi, Marchesoni, and Grigolini [26] for the overdamped colored-noise case into the formula for the MFPT proposed by Mel'nikov and Meshkov [15,20] for the damped white-noise case. The *ad hoc* formula reads

$$T_{\rm bot}=\frac{2\pi\,\exp\left(\dfrac{\Delta U}{k_BT}\right)A\left[\dfrac{\gamma(s_1+s_2)}{T}\right]\left[\dfrac{1+2\tau}{1-\tau}\right]^{1/2}}{\Omega_1\left[\left(1+\dfrac{\gamma^2}{4\omega^2}\right)^{1/2}-\dfrac{\gamma}{2\omega}\right]A\left(\dfrac{\gamma s_1}{T}\right)A\left[\dfrac{\gamma s_2}{T}\right]}, \qquad (6)$$

where Ω_1, ω, $A(\gamma s_1/T)$, $A(\gamma s_2/T)$, and $A[\gamma(s_1+s_2)/T]$ are explained in Eq. (2.72) of Ref. [15]. By this procedure, we hopefully account for the effects of the color of the noise as well as the damping term on the MFPT in a single formula. In Fig. 8 we compare our simulation results with the results calculated by this *ad hoc* formula. It is observed from Fig. 8 that our simulation results and the results calculated by the *ad hoc* formula do not agree satisfactorily. In fact, for a weak damping, the *ad hoc* formula overestimates our simulation results, whereas for a large damping coefficient, the *ad hoc* formula underestimates our simulation results.

V. CONCLUSIONS

This paper has concentrated on digital simulation of a damped bistable potential driven by white as well as OU noise. For the white-noise case, we have compared our digital simulation results for the MFPT with those of earlier theories and MCF results. It is found that theoretical and MCF results underestimate $T_{\rm bot}$ in the moderate to strong γ region. We have also measured the SPDF for the damped

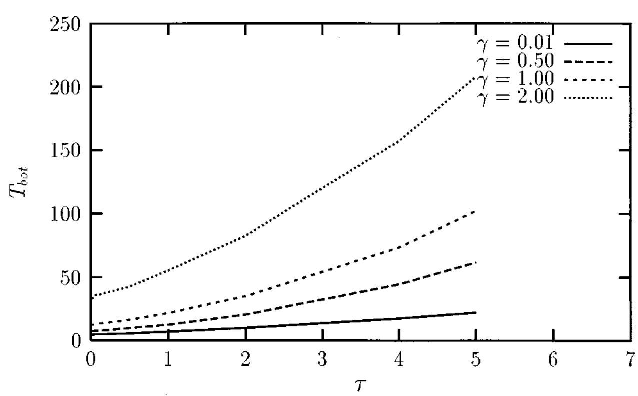

FIG. 5. $T_{\rm bot}$ as a function of τ for the damped bistable potential driven by OU noise with $D=0.5$ for various γ.

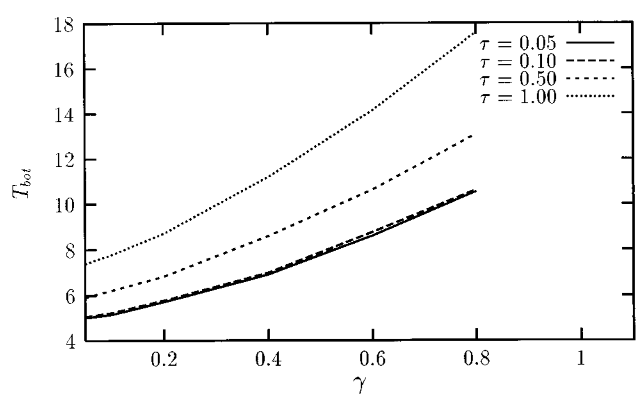

FIG. 7. $T_{\rm bot}$ vs γ for the damped bistable potential driven by OU noise with $D=0.5$ for various τ in the weak damping limit.

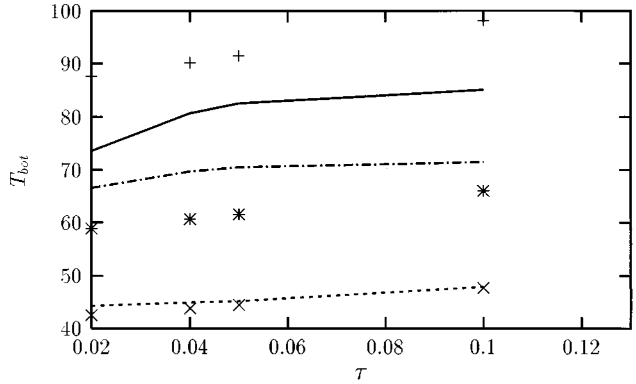

FIG. 8. $T_{\rm bot}$ vs τ for the damped bistable potential driven by OU noise for various γ. The results of the *ad hoc* formula (6) are compared with our simulation. Lines represent our simulation results with $D=0.04$ and $\gamma=0.4$ (—), $D=0.3$ and $\gamma=2.0$ (–·–·), and $D=0.15$ and $\gamma=1.0$ (----). Symbols represent the results of the *ad hoc* formula with $D=0.04$ and $\gamma=0.4$ (+), $D=0.3$ and $\gamma=2.0$ (*), and $D=0.15$ and $\gamma=1.0$ (×).

bistable system driven by OU noise using digital simulation and found it to agree with the corresponding analog simulation results closely.

We have computed the MFPT of a damped bistable potential driven by OU noise. We found that an increase in τ as well as an increase in γ has qualitatively the same effect on the MFPT. In the absence of a theoretical formula to compare our simulation results with, we use an *ad hoc* formula combining the formula for the MFPT proposed for the overdamped colored noise case by Hanggi, Marchesoni, and Grigolini [26] with the formula proposed for the MFPT for the damped white noise case by Mel'nikov and Meshkov [15,20]. However, this *ad hoc* formula fails to represent the behavior of $T_{\rm bot}$ vs τ in the whole damping range.

Currently we are attempting to develop the theory of a damped colored-noise-driven nonlinear stochastic system using the path integral technique [34]. The path integral method has proved to be a good technique for calculating the MFPT and the SPDF in the case of an overdamped bistable potential driven by OU noise [35,36]. We expect our simulation results to act as a benchmark for comparing future theories on a damped colored-noise-driven nonlinear stochastic system.

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